-*Discrete Mathematics

I- MCA / III- CS & IS

LECTURE NOTES

(B. E OF VTU)

VTU-EDUSAT Programme-17

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DISCRETE MATHEMATICS

Content

	CHAPTER
UNIT III	PROPERTIES OF INTEGERS

Introduction:

Mathematical induction is a method of mathematical proof typically used to establish that a given statement is true for all natural numbers (positive integers). It is done by proving that the first statement in the infinite sequence of statements is true, and then proving that if any one statement in the infinite sequence of statements is true, then so is the next one

The method can be extended to prove statements about more general well-founded structures, such as trees; this generalization, known as structural induction, is used in mathematical logic and computer science. Mathematical induction in this extended sense is closely related to recursion.

Mathematical induction should not be misconstrued as a form of inductive reasoning, which is considered non-rigorous in mathematics. In fact, mathematical induction is a form of rigorous deductive reasoning.

None of these ancient mathematicians, however, explicitly stated the inductive hypothesis. Another similar case was that of Francesco Maurolico in his *Arithmeticorum libri duo* (1575), is used the technique to prove that the sum of the first n odd integers is n^2. The first explicit formulation of the principle of induction was given by Pascal in his *Traité du triangle arithmétique* (1665). Another Frenchman, Fermat, made ample use of a related principle, indirect proof by infinite descent. The inductive hypothesis was also employed by the Swiss Jakob Bernoulli, and from then on it became more or less well known. The modern rigorous and systematic treatment of the principle came only in the 19th century, with George Boole, Augustus de Morgan, Charles Sanders Peirce, Giuseppe Peano, and Richard Dedekind.

“Mathematical induction" is unfortunately named, for it is unambiguously a form of deduction. However, it has certain similarities to induction which very likely inspired its name. It is like induction in that it generalizes to a whole class from a smaller sample. In fact, the sample is usually a sample of *one*, and the class is usually *infinite*. Mathematical induction is deductive, however, because the sample plus a rule about the unexamined cases actually gives us information about every member of the class. Hence the conclusion of a mathematical induction does not contain more information than was latent in the premises. Mathematical inductions therefore conclude with deductive certainty.

Mathematical induction is used frequently in discrete math and computer science. Many quantities that we are interested in measuring, such as running time, space, or output of a program, typically are restricted to positive integers, and thus mathematical induction is a natural way to prove facts about these quantities.

An analogy of mathematical induction is the game of dominoes. Suppose the dominoes are lined up properly, so that when one falls the successive one will also fall, Now one by pushing the first domino, the second will fall; when the second will fall; the third will fall; and so on. We can see that all dominoes will ultimately fall.

Objective

- Learn about Mathematical inductions
- Learn about recurrence relations
- Learn the relationship between sequences and recurrence relations
- Explore how to solve recurrence relations by iteration

Basic Introduction:

SEQUENCE:

A sequence is just a list of elements usually written in a row.

EXAMPLES:

1. 1, 2, 3, 4, 5, …
2. 1, 1/2, 1/3, 1/4, 1/5, …
3. 1, -1, 1, -1, 1, -1, …

FORMAL DEFINITION:

A sequence is a function whose domain is the set of integers greater than or equal to a particular integer n_0. Usually this set is the set of Natural numbers {1, 2, 3, …} or the set of whole numbers {0, 1, 2, 3, …}.

NOTATION:

We use the notation a_n to denote the image of the integer n, and call it a term of the sequence. Thus

$$a_1, a_2, a_3, a_4 \ldots, a_n, \ldots$$

represent the terms of a sequence defined on the set of natural numbers N.

Note that a sequence is described by listing the terms of the sequence in order of increasing subscripts.

FINDING TERMS OF A SEQUENCE GIVEN BY AN EXPLICIT FORMULA:

An explicit formula or general formula for a sequence is a rule that shows how the values of a_k depends on k.

EXAMPLE:

Define a sequence $a_1, a_2, a_3, \ldots$ by the explicit formula

$$a_k = \frac{k}{k+1} \qquad \text{for all integers } k \geq 1$$

The first four terms of the sequence are:

$$a_1 = \frac{1}{1+1} = \frac{1}{2}, a_2 = \frac{2}{2+1} = \frac{2}{3}, a_3 = \frac{3}{3+1} = \frac{3}{4}$$
$$and \quad fourth\ term\ is\ a_4 = \frac{4}{4+1} = \frac{4}{5}$$

EXAMPLE:

Write the first four terms of the sequence defined by the formula

$bj = 1 + 2^j$, for all integers $j \geq 0$

SOLUTION:

$b_0 = 1 + 2^0 = 1 + 1 = 2$

$b_1 = 1 + 2^1 = 1 + 2 = 3$

$b_2 = 1 + 2^2 = 1 + 4 = 5$

$b_3 = 1 + 2^3 = 1 + 8 = 9$

EXERCISE:

Compute the first six terms of the sequence defined by the formula $C_n = 1 + (-1)^n$ for all integers $n \geq 0$

SOLUTION :

$C_0 = 1 + (-1)^0 = 1 + 1 = 2$ $\qquad$ $C_1 = 1 + (-1)^1 = 1 + (-1) = 0$

$C_2 = 1 + (-1)^2 = 1 + 1 = 2$ $\qquad$ $C_3 = 1 + (-1)^3 = 1 + (-1) = 0$

$C_4 = 1 + (-1)^4 = 1 + 1 = 2$ $\qquad$ $C_5 = 1 + (-1)^5 = 1 + (-1) = 0$

EXAMPLE:

Write the first four terms of the sequence defined by

$$C_n = \frac{(-1)^n n}{n+1} \qquad \text{for all integers } n \geq 1$$

SOLUTION:

$$C_1 = \frac{(-1)^1(1)}{1+1} = \frac{-1}{2}, C_2 = \frac{(-1)^2(2)}{2+1} = \frac{2}{3}, C_3 = \frac{(-1)^3(3)}{3+1} = \frac{-3}{4}$$
$$And \quad fourth \quad term\ is C_4 = \frac{(-1)^4(4)}{4+1} = \frac{4}{5}$$

Note : A sequence whose terms alternate in sign is called an alternating sequence.

EXERCISES:

Find explicit formulas for sequences with the initial terms given:

1. 0, 1, -2, 3, -4, 5, …

SOLUTION: $a_n = (-1)^{n+1} n$ for all integers $n \geq 0$

2. $1-\frac{1}{2}, \frac{1}{2}-\frac{1}{3}, \frac{1}{3}-\frac{1}{4}, \frac{1}{4}-\frac{1}{5}, \ldots$

SOLUTION:

$$b_k = \frac{1}{k} - \frac{1}{k+1} \quad \text{for all integers } n \geq 1$$

3. 2, 6, 12, 20, 30, 42, 56, …

SOLUTION: $C_n = n(n + 1)$ for all integers $n \geq 1$

4. 1/4, 2/9, 3/16, 4/25, 5/36, 6/49, …

SOLUTION:

$$d_i = \frac{i}{(i+1)^2} \quad \text{for all integers } i \geq 1$$

OR

$$d_j = \frac{j+1}{(j+2)^2} \quad \text{for all integers } j \geq 0$$

ARITHMETIC SEQUENCE:

A sequence in which every term after the first is obtained from the preceding term by adding a constant number is called an arithmetic sequence or arithmetic progression (A.P.)

The constant number, being the difference of any two consecutive terms is called the common difference of A.P., commonly denoted by "d".

EXAMPLES:

1. 5, 9, 13, 17, … (common difference = 4)
2. 0, -5, -10, -15, … (common difference = -5)

3. x + a, x + 3a, x + 5a, … (common difference = 2a)

GENERAL TERM OF AN ARITHMETIC SEQUENCE:

Let **a** be the first term and **d** be the common difference of an arithmetic sequence. Then the sequence is a, a+d, a+2d, a+3d, …

If a_i, for $i \geq 1$, represents the terms of the sequence then a_n = nth term = a + (n - 1)d for all integers $n \geq 1$.

EXAMPLE: Find the 20th term of the arithmetic sequence 3, 9, 15, 21, …

SOLUTION:

Here a = first term = 3

d = common difference = 9 - 3 = 6

n = term number = 20

a_{20} = value of 20th term = ?

Since $a_n = a + (n - 1) d$; $n \geq 1$

$\therefore$ $a_{20} = 3 + (20 - 1) 6$

$= 3 + 114$

$= 117$

GEOMETRIC SEQUENCE:

A sequence in which every term after the first is obtained from the preceding term by multiplying it with a constant number is called a geometric sequence or geometric progression (G.P.)

The constant number, being the ratio of any two consecutive terms is called the common ratio of the G.P. commonly denoted by "r".

EXAMPLE:

1. 1, 2, 4, 8, 16, … (common ratio = 2)
2. 3, - 3/2, 3/4, - 3/8, … (common ratio = - 1/2)
3. 0.1, 0.01, 0.001, 0.0001, … (common ratio = 0.1 = 1/10)

GENERAL TERM OF A GEOMETRIC SEQUENCE:

Let **a** be the first tem and **r** be the common ratio of a geometric sequence. Then the sequence is a, ar, ar^2, ar^3, …

If a_i, for $i \geq 1$ represent the terms of the sequence, then

$$a_n = \text{nth term} = ar^{n-1}; \quad \text{for all integers } n \geq 1$$

EXAMPLE:

Write the geometric sequence with positive terms whose second term is 9 and fourth term is 1.

SOLUTION: Let **a** be the first term and **r** be the common ratio of the geometric sequence. Then

$a_n = ar^{n-1} \qquad n \geq 1$

Now $a_2 = ar^{2-1}$

$\Rightarrow \quad 9 = ar$…………………(1)

Also $a_4 = ar^{4-1}$

$1 = ar^3$…………………(2)

Dividing (2) by (1), we get,

$$\frac{1}{9} = \frac{ar^3}{ar}$$

$$\Rightarrow \quad \frac{1}{9} = r^2$$

$$\Rightarrow \quad r = \frac{1}{3} \qquad \left(\text{rejecting } r = -\frac{1}{3}\right)$$

Substituting r = 1/3 in (1), we get

$$9 = a\left(\frac{1}{3}\right)$$

$$\Rightarrow \quad a = 9 \times 3 = 27$$

Hence the geometric sequence is

27, 9, 3, 1, 1/3, 1/9, …

SEQUENCES IN COMPUTER PROGRAMMING:

An important data type in computer programming consists of finite sequences known as one-dimensional arrays; a single variable in which a sequence of variables may be stored.

EXAMPLE:

The names of k students in a class may be represented by an array of k elements “name” as:

name [0], name[1], name[2], …, name[k-1]

SERIES:

The sum of the terms of a sequence forms a series. If $a_1, a_2, a_3, \ldots$ represent a sequence of numbers, then the corresponding series is

$$a_1 + a_2 + a_3 + \ldots = \sum_{k=1}^{\infty} a_k$$

SUMMATION NOTATION:

The capital Greek letter sigma $\sum$ is used to write a sum in a short hand notation.

More generally if m and n are integers and $m \le n$, then the summation from k equal m to n of a_k is

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

Here **k** is called the index of the summation; **m** the lower limit of the summation and **n** the upper limit of the summation.

COMPUTING SUMMATIONS:

Let $a_0 = 2$, $a_1 = 3$, $a_2 = -2$, $a_3 = 1$ and $a_4 = 0$.Compute each of the summations:

1. $\sum_{i=0}^{4} a_i$

SOLUTION:

$$\sum_{i=0}^{4} a_i = a_0 + a_1 + a_2 + a_3 + a_4$$

$$= 2 + 3 + (-2) + 1 + 0 = 4$$

SUMMATION NOTATION TO EXPANDED FORM:

$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1}$$

EXPANDED FORM TO SUMMATION NOTATION:

Write the following using summation notation:

$$\frac{1}{n}+\frac{2}{n+1}+\frac{3}{n+2}+\cdots+\frac{n+1}{2n}$$

ARITHMETIC SERIES:

In general, if a is the first term and d the common difference of an arithmetic series, then the series is given as: a + (a+d) + (a+2d) +…

SUM OF n TERMS OF AN ARITHMETIC SERIES:

Let a be the first term and d be the common difference of an arithmetic series. Then its nth term is:

$$a_n = a + (n - 1)d; \qquad n \geq 1$$

If S_n denotes the sum of first n terms of the A.S, then $S_n = n(a + a_n)/2$

$$S_n = n(a + l)/2 \qquad \text{Where} \qquad l = an = a + (n - 1)d$$

Therefore

$$S_n = n/2\ [2\ a + (n - 1)\ d$$

GEOMETRIC SERIES:

If **a** is the first term and **r** the common ratio of a geometric series, then the series is given as: a + ar + ar^2 + ar^3 + …

SUM OF n TERMS OF A GEOMETRIC SERIES:

Let **a** be the first term and **r** be the common ratio of a geometric series. Then its nth term is:

$$a_n = ar^{n-1}; \qquad n \geq 1$$

$$\Rightarrow \qquad S_n = \frac{a(1-r^n)}{1-r} \qquad (r \neq 1)$$

INFINITE GEOMETRIC SERIES:

Consider the infinite geometric series

$a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$

then

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r} \qquad (r \neq 1)$$

If $S_n \to S$ as $n \to \infty$, then the series is convergent and S is its sum.

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$

$$\therefore \qquad S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$$

If S_n increases indefinitely as n becomes very large then the series is said to be divergent.

IMPORTANT SUMS:

1. $1+2+3+\cdots+n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
2. $1^2+2^2+3^2+\cdots+n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
3. $1^3+2^3+3^3+\cdots+n^3 = \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)}{4} = \left[\frac{n(n+1)}{2}\right]^2$

Recursion is a principle closely related to mathematical induction. In a **recursive definition**, an object is defined in terms of itself. We can recursively define **sequences**, **functions** and **sets**.

Example:

The sequence $\{a_n\}$ of powers of 2 is given by $a_n = 2^n$ for $n = 0, 1, 2, \dots$.

The same sequence can also be defined **recursively**: $a_0 = 1$

$a_{n+1} = 2a_n$ for $n = 0, 1, 2, \dots$

Obviously, induction and recursion are similar principles.

We can use the following method to define a function with the **natural numbers** as its domain:

1. Specify the value of the function at zero.
2. Give a rule for finding its value at any integer from its values at smaller integers.

Such a definition is called **recursive** or **inductive definition**.

How can we recursively define the factorial function f(n) = n! ?

$f(0) = 1$

$f(n + 1) = (n + 1)f(n)$

$f(0) = 1$

$f(1) = 1f(0) = 1 \cdot 1 = 1$

$f(2) = 2f(1) = 2 \cdot 1 = 2$

$f(3) = 3f(2) = 3 \cdot 2 = 6$

$f(4) = 4f(3) = 4 \cdot 6 = 24$

If we want to recursively define a set, we need to provide two things:

- an **initial set** of elements,
- **rules** for the construction of **additional** elements from elements in the set.

Example: Let S be recursively defined by: $3 \in S$, $(x + y) \in S$ if $(x \in S)$ and $(y \in S)$

S is the set of positive integers divisible by 3.

Let P(n) be the statement “3n belongs to S”.

Basis step: P(1) is true, because 3 is in S.

Inductive step: To show: If P(n) is true, then P(n + 1) is true.

Assume 3n is in S. Since 3n is in S and 3 is in S, it follows from the recursive definition of S that
$3n + 3 = 3(n + 1)$ is also in S.

First of all instead of giving the definition of Recursion we give you an example, you already know the Set of Odd numbers Here we give the new definition of the same set that is the set of Odd numbers.

Definition for odd positive integers may be given as:

BASE:

1 is an odd positive integer.

RECURSION:

If k is an odd positive integer, then k + 2 is an odd positive integer.

Now, 1 is an odd positive integer by the definition base.

With k = 1, 1 + 2 = 3, so 3 is an odd positive integer.

With k = 3, 3 + 2 = 5, so 5 is an odd positive integer

and so, 7, 9, 11, … are odd positive integers.

REMARK: Recursive definitions can be used in a "generative" manner.

RECURSION:

The process of defining an object in terms of smaller versions of itself is called recursion.

A recursive definition has two parts:

1.BASE:

An initial simple definition which **cannot** be expressed in terms of smaller versions of itself.

2. RECURSION:

The part of definition which **can** be expressed in terms of smaller versions of itself.

RECURSIVELY DEFINED FUNCTIONS:

A function is said to be recursively defined if the function refers to itself such that

1. There are certain arguments, called base values, for which the function does not refer to itself.

2. Each time the function does refer to itself, the argument of the function must be closer to a base value.

EXAMPLE:

Suppose that f is defined recursively by

$f(0) = 3$

$f(n + 1) = 2 f(n) + 3$

Find $f(1), f(2), f(3)$ and $f(4)$

SOLUTION:

From the recursive definition it follows that

$f(1) = 2 f(0) + 3 = 2(3) + 3 = 6 + 3 = 9$

In evaluating of f(1) we use the formula given in the example and we note that it involves f(0) and we are also given the value of that which we use to find out the functional value at 1. Similarly we will use the preceding value

In evaluating the next values of the functions as we did below.

$f(2) = 2 f(1) + 3 = 2(9) + 3 = 18 + 3 = 21$

$f(3) = 2 f(2) + 3 = 2(21) + 3 = 42 + 3 = 45$

$f(4) = 2 f(3) + 3 = 2(45) + 3 = 90 + 3 = 93$

EXERCISE:

Find f(2), f(3), and f(4) if f is defined recursively by

$f(0) = -1, f(1)=2$ and for $n = 1, 2, 3, \dots$

$f(n+1) = f(n) + 3 f(n - 1)$

SOLUTION:

From the recursive definition it follows that

$f(2) = f(1) + 3 f(1-1)$

$= f(1) + 3 f(0)$

$= 2 + 3(-1) \quad = -1$

Now in order to find out the other values we will need the values of the preceding .So we write these values here again

$f(0) = -1, f(1)=2 \qquad f(n+1) = f(n) + 3 f(n - 1)$

$f(2) = -1$

By recursive formula we have

$f(3) = f(2) + 3 f(2-1)$

$= f(2) + 3 f(1)$

$= (-1) + 3 (2)$

$= 5$

$f(4) = f(3) + 3 f(3-1)$

$= f(2) + 3 f(2)$

$= 5 + 3 (-1)$

$= 2$

THE FACTORIAL OF A POSITIVE INTEGER:

For each positive integer n, the factorial of n denoted n! is defined to be the product of all the integers from 1 to n:

$n! = n\cdot(n - 1)\cdot(n - 2) \cdot\cdot\cdot 3 \cdot 2 \cdot 1$

Zero factorial is defined to be 1

$0! = 1$

In general,

$n! = n(n-1)!$ for each positive integer n.

THE FACTORIAL FUNCTION DEFINED RECURSIVELY:

We can define the factorial function F(n) = n! recursively by specifying the initial value of this function, namely, F(0) = 1, and giving a rule for finding F(n) from F(n-1).{(n! = n(n-1)!}

Thus, the recursive definition of factorial function F(n) is:

1. F(0) = 1
2. F(n) = n F(n-1)

EXERCISE:

Let S be the function such that S(n) is the sum of the first n positive integers. Give a recursive definition of S(n).

SOLUTION:

The initial value of this function may be specified as S(0) = 0

Since

$$S(n) = n + (n - 1) + (n - 2) + \dots + 3 + 2 + 1$$
$$= n + [(n - 1) + (n - 2) + \dots + 3 + 2 + 1]$$
$$= n + S(n-1)$$

which defines the recursive step.

Accordingly S may be defined as:

1. S(0)= 0
2. S(n) = n + S(n - 1) for n ≥ 1

EXERCISE:

Let a and b denote positive integers. Suppose a function Q is defined recursively as follows:

(a) Find the value of Q(2,3) and Q(14,3)

(b) What does this function do? Find Q (3355, 7)

SOLUTION:

$$Q(a,b) = \begin{cases} 0 & \text{if } a \langle b \\ Q(a-b,b)+1 & \text{if } b \le a \end{cases}$$

(a) Q (2,3) = 0 since 2 < 3

Given Q(a,b) = Q(a-b,b) + 1 if b ≤a

Now

Q (14, 3) = Q (11,3) + 1

= [Q(8,3) + 1] + 1 = Q(8,3) + 2

= [Q(5,3) + 1] + 2 = Q(5,3) + 3

= [Q(2,3) + 1] + 3 = Q(2,3) + 4

= 0 + 4 (∴ Q(2,3) = 0)

= 4

(b)

$$Q(a,b) = \begin{cases} 0 & \text{if } a \langle b \\ Q(a-b,b)+1 & \text{if } b \le a \end{cases}$$

Each time b is subtracted from a, the value of Q is increased by 1. Hence Q(a,b) finds the integer quotient when a is divided by b.

Thus Q(3355, 7) = 479

THE FIBONACCI SEQUENCE:

The Fibonacci sequence is defined as follows.

$F_0 = 1, F_1 = 1$

$F_k = F_{k-1} + F_{k-2}$ for all integers $k \ge 2$

$F_2 = F_1 + F_0 = 1 + 1 = 2$

$F_3 = F_2 + F_1 = 2 + 1 = 3$

$F_4 = F_3 + F_2 = 3 + 2 = 5$

$F_5 = F_4 + F_3 = 5 + 3 = 8$

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RECURRENCE RELATION:

A recurrence relation for a sequence $a_0, a_1, a_2, \ldots,$ is a formula that relates each term a_k to certain of its predecessors $a_{k-1}, a_{k-2}, \ldots, a_{k-i}$,

where i is a fixed integer and k is any integer greater than or equal to i. The initial conditions for such a recurrence relation specify the values of

$a_0, a_1, a_2, \ldots, a_{i-1}$.

EXERCISE:

Find the first four terms of the following recursively defined sequence.

$b_1 = 2$

$b_k = b_{k-1} + 2 \cdot k,$ for all integers $k \ge 2$

SOLUTION:

$b_1 = 2$ (given in base step)

$b_2 = b_1 + 2 \cdot 2 = 2 + 4 = 6$

$b_3 = b_2 + 2 \cdot 3 = 6 + 6 = 12$

$b_4 = b_3 + 2 \cdot 4 = 12 + 8 = 20$

EXERCISE:

Find the first five terms of the following recursively defined sequence.

$t_0 = -1, \qquad t_1 = 1$

$t_k = t_{k-1} + 2 \cdot t_{k-2}, \qquad$ for all integers $k \geq 2$

SOLUTION:

$t_0 = -1,$ (given in base step)

$t_1 = 1$ (given in base step)

$t_2 = t_1 + 2 \cdot t_0 = 1 + 2 \cdot (-1) = 1 - 2 = -1$

$t_3 = t_2 + 2 \cdot t_1 = -1 + 2 \cdot 1 = -1 + 2 = 1$

$t_4 = t_3 + 2 \cdot t_2 = 1 + 2 \cdot (-1) = 1 - 2 = -1$

EXERCISE:

Define a sequence $b_0, b_1, b_2, \ldots$ by the formula

$b_n = 5^n, \quad$ for all integers $n \geq 0$.

Show that this sequence satisfies the recurrence relation $b_k = 5b_{k-1}$, for all integers $k \geq 1$.

SOLUTION:

The sequence is given by the formula

$$b_n = 5^n$$

Substituting k for n we get

$$b_k = 5^k \quad \ldots\ldots (1)$$

Substituting k – 1 for n we get

$$b_{k-1} = 5^{k-1} \quad \ldots\ldots (2)$$

Multiplying both sides of (2) by 5 we obtain

$$5 \cdot b_{k-1} = 5 \cdot 5^{k-1}$$

$$= 5^k = b_k \qquad \text{using (1)}$$

Hence $\quad b_k = 5b_{k-1} \quad$ as required

EXERCISE:

Show that the sequence 0, 1, 3, 7, . . . , $2^n - 1$, . . . , for $n \geq 0$, satisfies the recurrence relation

$$d_k = 3d_{k-1} - 2d_{k-2}, \text{ for all integers } k \geq 2$$

SOLUTION:

The sequence is given by the formula

$$d_n = 2^n - 1 \qquad \text{for } n \geq 0$$

Substituting k – 1 for n we get $d_{k-1} = 2^{k-1} - 1$

Substituting k – 2 for n we get $d_{k-2} = 2^{k-2} - 1$

We want to prove that

$$d_k = 3d_{k-1} - 2d_{k-2}$$

$$\text{R.H.S.} = 3(2^{k-1} - 1) - 2(2^{k-2} - 1)$$

$$= 3 \cdot 2^{k-1} - 3 - 2 \cdot 2^{k-2} + 2$$

$$= 3 \cdot 2^{k-1} - 2^{k-1} - 1$$

$$= (3 - 1) \cdot 2^{k-1} - 1$$

$$= 2 \cdot 2^{k-1} - 1 = 2^k - 1 = d_k \quad = \text{L.H.S.}$$

THE TOWER OF HANOI:

The puzzle was invented by a French Mathematician Adouard Lucas in 1883. It is well known to students of Computer Science since it appears in virtually any introductory text on data structures or algorithms.

There are three poles on first of which are stacked a number of disks that decrease in size as they rise from the base. The goal is to transfer all the disks one by one from the first pole to one of the others, but they must never place a larger disk on top of a smaller one.

Let m_n be the minimum number of moves needed to move a tower of n disks from one pole to another. Then m_n can be obtained recursively as follows.

- $m_1 = 1$
- $m_k = 2\, m_{k-1} + 1$

$m_2 = 2 \cdot m_1 + 1 = 2 \cdot 1 + 1 = 3$

$m_3 = 2 \cdot m_2 + 1 = 2 \cdot 3 + 1 = 7$

$m_4 = 2 \cdot m_3 + 1 = 2 \cdot 7 + 1 = 15$

$m_5 = 2 \cdot m_4 + 1 = 2 \cdot 15 + 1 = 31$

$m_6 = 2 \cdot m_5 + 1 = 2 \cdot 31 + 1 = 65$

Note that

$$m_n = 2^n - 1$$

$$m_{64} = 2^{64} - 1 \quad \cong 584.5 \text{ billion years}$$

USE OF RECURSION:

At first recursion may seem hard or impossible, may be magical at best. However, recursion often provides elegant, short algorithmic solutions to many problems in computer science and mathematics.

Examples where recursion is often used

- math functions
- number sequences
- data structure definitions
- data structure manipulations
- language definitions

PRINCIPLE OF MATHEMATICAL INDUCTION:

Let P(n) be a propositional function defined for all positive integers n. P(n) is true for every positive integer n if

1.Basis Step:

The proposition P(1) is true.

2.Inductive Step:

If P(k) is true then P(k + 1) is true for all integers k ≥ 1.

i.e. $\forall$ k p(k) → P(k + 1)

EXAMPLE:

Use Mathematical Induction to prove that

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$ for all integers n ≥1

SOLUTION:

Let $$P(n): 1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

1.Basis Step:

P(1) is true.

For n = 1, left hand side of P(1) is the sum of all the successive integers starting at 1 and ending at 1, so LHS = 1 and RHS is

$$R.H.S=\frac{1(1+1)}{2}=\frac{2}{2}=1$$

so the proposition is true for n = 1.

2. Inductive Step: Suppose P(k) is true for, some integers k ≥ 1.

(1) $$1+2+3+\cdots+k=\frac{k(k+1)}{2}$$

To prove P(k + 1) is true. That is,

$$(2)\qquad 1+2+3+\cdots+(k+1)=\frac{(k+1)(k+2)}{2}$$

Consider L.H.S. of (2)

$$\begin{aligned}1+2+3+\cdots+(k+1) &= 1+2+3+\cdots+k+(k+1)\\ &= \frac{k(k+1)}{2}+(k+1) \qquad \text{using (1)}\\ &= (k+1)\left[\frac{k}{2}+1\right]\\ &= (k+1)\left[\frac{k+2}{2}\right]\\ &= \frac{(k+1)(k+2)}{2} = \text{RHS of (2)}\end{aligned}$$

Hence by principle of Mathematical Induction the given result true for all integers greater or equal to 1.

EXERCISE:

Use mathematical induction to prove that

$1+3+5+\ldots+(2n-1) = n^2$ for all integers $n \geq 1$.

SOLUTION:

Let P(n) be the equation $1+3+5+\ldots+(2n-1) = n^2$

1. **Basis Step**:

   P(1) is true

   For n = 1, L.H.S of P(1) = 1and

   R.H.S =2(1)-1 = 1

   Hence the equation is true for n = 1

2. **Inductive Step**:

   Suppose P(k) is true for some integer k ≥ 1. That is,

   $1 + 3 + 5 + \ldots + (2k - 1) = k^2$ …………………(1)

To prove P(k+1) is true; i.e.,

$1 + 3 + 5 + \ldots + [2(k+1)-1] = (k+1)^2$ ………….……(2)

Consider L.H.S. of (2)

$$\begin{aligned} 1+3+5+\cdots+[2(k+1)-1] &= 1+3+5+\cdots+(2k+1) \\ &= 1+3+5+\cdots+(2k-1)+(2k+1) \\ &= k^2+(2k+1) \qquad \text{using (1)} \\ &= (k+1)^2 \\ &= \text{R.H.S. of (2)} \end{aligned}$$

Thus P(k+1) is also true. Hence by mathematical induction, the given equation is true for all integers n≥1.

EXERCISE:

Use mathematical induction to prove that

$1+2+2^2 + \ldots + 2^n = 2^{n+1} - 1$ for all integers $n \geq 0$

SOLUTION:

Let P(n): $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$

1. **Basis Step:**

P(0) is true.

For n = 0

L.H.S of P(0) = 1

R.H.S of P(0) = $2^{0+1} - 1 = 2 - 1 = 1$

Hence P(0) is true.

2. **Inductive Step:**

Suppose P(k) is true for some integer k ≥ 0; i.e.,

$1+2+2^2+\ldots+2^k = 2^{k+1} - 1$……………………(1)

To prove P(k+1) is true, i.e.,

$1+2+2^2+\ldots+2^{k+1} = 2^{k+1+1} - 1$…………………(2)

Consider LHS of equation (2)

$1+2+2^2+\ldots+2^{k+1}= (1+2+2^2+\ldots+2^k) + 2^{k+1}$

$= (2^{k+1} - 1) +2^{k+1}$

$= 2\cdot2^{k+1} - 1$

$= 2^{k+1+1} - 1$ = R.H.S of (2)

Hence P(k+1) is true and consequently by mathematical induction the given propositional function is true for all integers n≥0.

EXERCISE:

Prove by mathematical induction

$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$ for all integers n ≥1.

SOLUTION:

Let P(n) denotes the given equation

1. **Basis step:**

P(1) is true

For n = 1

L.H.S of P(1) = $1^2 = 1$

R.H.S of P(1) $=\frac{1(1+1)(2(1)+1)}{6}$

$=\frac{(1)(2)(3)}{6}=\frac{6}{6}=1$

So L.H.S = R.H.S of P(1).Hence P(1) is true

2.Inductive Step:

Suppose P(k) is true for some integer k ≥1;

$$1^2+2^2+3^2+\cdots+k^2=\frac{k(k+1)(2k+1)}{6} \quad \ldots\ldots\ldots(1)$$

To prove P(k+1) is true; i.e.;

$$1^2+2^2+3^2+\cdots+(k+1)^2=\frac{(k+1)(k+1+1)(2(k+1)+1)}{6} \quad \ldots\ldots\ldots(2)$$

Consider LHS of above equation (2)

$$1^2+2^2+3^2+\cdots+(k+1)^2 \quad =1^2+2^2+3^2+\cdots+k^2+(k+1)^2$$

$$=\frac{k(k+1)(2k+1)}{6}+(k+1)^2 \quad =(k+1)\left[\frac{k(2k+1)+6(k+1)}{6}\right]$$

$$=(k+1)\left[\frac{2k^2+k+6k+6}{6}\right]$$

$$=\frac{(k+1)(2k^2+7k+6)}{6}$$

$$=\frac{(k+1)(k+2)(2k+3)}{6}$$

$$=\frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$$

$$=(k+1)\left[\frac{2k^2+k+6k+6}{6}\right]$$

$$=\frac{(k+1)(2k^2+7k+6)}{6}$$

$$=\frac{(k+1)(k+2)(2k+3)}{6}$$

$$=\frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$$

EXERCISE:

Prove by mathematical induction

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$ for all integers n≥1

SOLUTION:

Let P(n) be the given equation.

1.Basis Step: P(1) is true

For n = 1

L.H.S of P(1) = $\frac{1}{1\cdot 2}=\frac{1}{1\times 2}=\frac{1}{2}$

R.H.S of P(1) = $\frac{1}{1+1}=\frac{1}{2}$

Hence P(1) is true

2.Inductive Step:

Suppose P(k) is true, for some integer k≥1. That is

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1} \qquad \text{……….(1)}$$

To prove P(k+1) is true. That is

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)(k+1+1)}=\frac{k+1}{(k+1)+1} \qquad \text{……..(2)}$$

Now we will consider the L.H.S of the equation (2) and will try to get the R.H.S by using equation (1) and some simple computation.

Consider LHS of (2)

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)(k+2)}$$
$$=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$
$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$
$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$
$$=\frac{(k+1)^2}{(k+1)(k+2)}$$
$$=\frac{k+1}{(k+2)}$$
$$=\text{RHS of (2)}$$

Hence P(k+1) is also true and so by Mathematical induction the given equation is true for all integers n ≥1.

EXERCISE:

Use mathematical induction to prove that

$$\sum_{i=1}^{n+1} i2^i = n\cdot 2^{n+2} + 2, \qquad \text{for all integers } n \ge 0$$

SOLUTION:

1.Basis Step:

To prove the formula for n = 0, we need to show that

$$\sum_{i=1}^{0+1} i.2^i = 0\cdot 2^{0+2} + 2$$

Now, L.H.S = $\sum_{i=1}^{1} i\cdot 2^i = (1)2^1 = 2$

R.H.S = $0\cdot 2^2 + 2 = 0 + 2 = 2$

Hence the formula is true for n = 0

2.Inductive Step:

Suppose for some integer n=k ≥0

$$\sum_{i=1}^{k+1} i\cdot 2^i = k\cdot 2^{k+2} + 2 \qquad \text{..................(1)}$$

We must show that

$$\sum_{i=1}^{k+2} i\cdot 2^i = (k+1)\cdot 2^{k+1+2} + 2 \qquad \text{................(2)}$$

Consider LHS of (2)

$$\begin{aligned}
\sum_{i=1}^{k+2} i\cdot 2^i &= \sum_{i=1}^{k+1} i\cdot 2^i + (k+2)\cdot 2^{k+2} \\
&= (k\cdot 2^{k+2} + 2) + (k+2)\cdot 2^{k+2} \\
&= (k+k+2)2^{k+2} + 2 \\
&= (2k+2)\cdot 2^{k+2} + 2 \\
&= (k+1)2\cdot 2^{k+2} + 2 \\
&= (k+1)\cdot 2^{k+1+2} + 2 \\
&= \text{RHS of equation (2)}
\end{aligned}$$

Hence the inductive step is proved as well. Accordingly by mathematical induction the given formula is true for all integers n≥0.

EXERCISE:

Use mathematical induction to prove that

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$$

for all integers n ≥2

SOLUTION:

1. **Basis Step**:

For n = 2

L.H.S $=1-\frac{1}{2^2}=1-\frac{1}{4}=\frac{3}{4}$

R.H.S $=\frac{2+1}{2(2)}=\frac{3}{4}$

Hence the given formula is true for n = 2

2. **Inductive Step**:

Suppose for some integer k ≥2

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

We must show that

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)=\frac{(k+1)+1}{2(k+1)} \quad \ldots\ldots\ldots\ldots..(2)$$

Consider L.H.S of (2)

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\left[\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\right]\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\left(\frac{k+1}{2k}\right)\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right)$$

$$=\left(\frac{1}{2k}\right)\left(\frac{k^2+2k+1-1}{(k+1)}\right)$$

$$= \frac{k^2 + 2k}{2k(k+1)} = \frac{k(k+2)}{2k(k+1)}$$

$$= \frac{k+1+1}{2(k+1)} = \quad \text{RHS of (2)}$$

Hence by mathematical induction the given equation is true

EXERCISE: Prove by mathematical induction

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1$$

for all integers n≥1

SOLUTION:

1.Basis step:

For n = 1

L.H.S $= \sum_{i=1}^{n} i(i!) = (1)(1!) = 1$

R.H.,S = (1+1)! - 1 = 2! - 1

= 2 -1 = 1

Hence

$$\sum_{i=1}^{1} i(i!) = (1+1)! - 1$$

which proves the basis step.

2.Inductive Step: Suppose for any integer k ≥1

$$\sum_{i=1}^{k} i(i!) = (k+1)! - 1 \qquad \text{..............................(1)}$$

We need to prove that

$$\sum_{i=1}^{k+1} i(i!) = (k+1+1)! - 1 \qquad \text{.........................(2)}$$

Consider LHS of (2) $\sum_{i=1}^{k+1} i(i!) = \sum_{i=1}^{k} i(i!) + (k+1)(k+1)!$ Using (1)

$$= (k+1)! - 1 + (k+1)(k+1)!$$
$$= (k+1)! + (k+1)(k+1)! - 1$$
$$= [1 + (k+1)](k+1)! - 1$$
$$= (k+2)(k+1)! - 1$$
$$= (k+2)! - 1$$
$$= \text{RHS of (2)}$$

Hence the inductive step is also true.Accordingly, by mathematical induction, the given formula is true for all integers n ≥1.

EXERCISE: Use mathematical induction to prove the generalization of the following DeMorgan's Law:

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

where $A_1, A_2, \ldots, A_n$ are subsets of a universal set U and n≥2.

SOLUTION:

Let P(n) be the given propositional function

1.Basis Step: P(2) is true.

$$\text{L.H.S of } P(2) = \overline{\bigcap_{j=1}^{2} A_j} = \overline{A_1 \bigcap A_2}$$

$$= \overline{A_1} \bigcup \overline{A_2} \quad \text{By DeMorgan's Law}$$

$$= \bigcup_{i=1}^{2} \overline{A_j} = \text{RHS of } P(2)$$

2.Inductive Step:

Assume that P(k) is true for some integer k ≥2; i.e.,

$$\overline{\bigcap_{j=1}^{k} A_j} = \bigcup_{j=1}^{k} \overline{A_j} \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

where $A_1, A_2, \ldots, A_k$ are subsets of the universal set U. If A_{k+1} is another set of U, then we need to show that

$$\overline{\bigcap_{j=1}^{k+1} A_j} = \bigcup_{j=1}^{k+1} \overline{A_j} \qquad \ldots\ldots\ldots\ldots\ldots(2)$$

Consider L.H.S of (2)

$$\overline{\bigcap_{j=1}^{k+1} A_j} = \overline{\left(\bigcap_{j=1}^{k} A_j\right) \cap A_{k+1}}$$

$$= \overline{\left(\bigcap_{j=1}^{k} A_j\right)} \cup \overline{A_{k+1}}$$

$$= \left(\bigcup_{j=1}^{k} \overline{A_j}\right) \cup \overline{A_{k+1}}$$

$$= \bigcup_{j=1}^{k+1} \overline{A_j} \qquad \text{By DeMorgan's Law}$$

$$= \text{R.H.S of } (2)$$

Hence by mathematical induction, the given generalization of DeMorgan's Law holds.

MATHEMATICAL INDUCTION FOR DIVISIBILITY PROBLEMS AND INEQUALITY PROBLEMS

DIVISIBILITY:

Let n and d be integers and $d \neq 0$. Then n is divisible by d or d divides n written d|n. iff $n = d \cdot k$ for some integer k.

Alternatively, we say that n is a multiple of d, d is a divisor of n , d is a factor of n

Thus $d|n \Leftrightarrow \exists$ an integer k such that $n = d \cdot k$

EXERCISE:

Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

SOLUTION:

1. Basis Step:

For n = 1

$$n^3 - n = 1^3 - 1 = 1 - 1 = 0$$

which is clearly divisible by 3, since $0 = 0 \cdot 3$

Therefore, the given statement is true for n = 1.

2.Inductive Step:

Suppose that the statement is true for n = k, i.e., k^3-k is divisible by 3 for all $n \in Z+$

Then

$$k^3-k = 3 \cdot q \text{……………………….(1)}$$

for some $q \in Z$

We need to prove that $(k+1)^3 - (k+1)$ is divisible by 3.

Now

$$(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k + 1)$$

$$= k^3 + 3k^2 + 2k$$

$$= (k^3-k) + 3k^2 + 2k + k$$

$= (k^3 - k) + 3k^2 + 3k$

$= 3 \cdot q + 3 \cdot (k^2 + k)$ using(1)

$= 3[q + k^2 + k]$

$\Rightarrow$ $(k+1)^3 - (k+1)$ is divisible by 3.

Hence by mathematical induction $n^3 - n$ is divisible by 3, whenever n is a positive integer.

EXAMPLE:

Use mathematical induction to prove that for all integers $n \geq 1$,

$2^{2n} - 1$ is divisible by 3.

SOLUTION:

Let P(n): $2^{2n} - 1$ is divisible by 3.

1.Basis Step:

P(1) is true

Now P(1): $2^{2(1)} - 1$ is divisible by 3.

Since $2^{2(1)} - 1 = 4 - 1 = 3$

which is divisible by 3.

Hence P(1) is true.

2.Inductive Step:

Suppose that P(k) is true. That is $2^{2k} - 1$ is divisible by 3. Then, there exists an integer q such that

$2^{2k} - 1 = 3 \cdot q$ ....(1)

To prove P(k+1) is true, that is $2^{2(k+1)} - 1$ is divisible by 3.

Now consider

$2^{2(k+1)} - 1 = 2^{2k+2} - 1$

$= 2^{2k} 2^2 - 1$

$= 2^{2k} 4 - 1$

$= 2^{2k}(3+1) - 1$

$= 2^{2k}\cdot 3+(2^{2k} - 1)$

$= 2^{2k}\cdot 3+3\cdot q$ [by using (1)]

$=3(2^{2k} + q)$

$\Rightarrow 2^{2(k+1)} - 1$ is divisible by 3.

Accordingly, by mathematical induction. $2^{2n}- 1$ is divisible by 3, for all integers $n \geq 1$.

EXERCISE:

Use mathematical induction to show that the product of any two consecutive positive integers is divisible by 2.

SOLUTION:

Let n and n + 1 be two consecutive integers. We need to prove that n(n+1) is divisible by 2.

1. **Basis Step:** For n = 1

$n (n+1) = 1\cdot(1+1) = 1\cdot 2 = 2$

which is clearly divisible by 2.

2. **Inductive Step**:

Suppose the given statement is true for n = k. That is

k (k+1) is divisible by 2, for some $k \in Z+$

Then $k (k+1) = 2\cdot q$ ………………….(1) $q \in Z+$

We must show that

(k+1)(k+1+1) is divisible by 2.

Consider (k+1)(k+1+1) = (k+1)(k+2)

= (k+1)k + (k+1)2

= 2q + 2 (k+1) using (1) = 2(q+k+1)

Hence (k+1) (k+1+1) is also divisible by 2.

Accordingly, by mathematical induction, the product of any two consecutive positive integers is divisible by 2.

EXERCISE:

Prove by mathematical induction $n^3 - n$ is divisible by 6, for each integer $n \geq 2$.

SOLUTION:

1.Basis Step: For $n = 2$

$n^3 - n = 2^3 - 2 = 8 - 2 = 6$

which is clearly divisible by 6, since $6 = 1 \cdot 6$

Therefore, the given statement is true for $n = 2$.

2.Inductive Step:

Suppose that the statement is true for $n = k$, i.e., $k^3 - k$ is divisible by 6, for all integers $k \geq 2$.

Then

$k^3 - k = 6 \cdot q$……………(1) for some $q \in Z$.

We need to prove that

$(k+1)^3 - (k+1)$ is divisible by 6

Now $(k+1)^3 - (k+1) = (k^3 + 3k^3 + 3k + 1) - (k+1)$

$= k^3 + 3k^3 + 2k$

$= (k^3 - k) + (3k^3 + 2k + k)$

$= (k^3 - k) + 3k^3 + 3k$ Using (1)

$= 6 \cdot q + 3k(k+1)$………………..(2)

Since k is an integer, so k(k+1) being the product of two consecutive integers is an even number.

Let $k(k+1) = 2r$ $r \in Z$

Now equation (2) can be rewritten as:

$(k+1)^3 - (k+1) = 6 \cdot q + 3 \cdot 2\, r$

$= 6q + 6r$

$= 6(q+r)$ $q, r \in Z$

$\Rightarrow (k+1)^3 - (k+1)$ is divisible by 6.

Hence, by mathematical induction, $n^3 - n$ is divisible by 6, for each integer $n \geq 2$.

EXERCISE:

Prove by mathematical induction. For any integer $n \geq 1$, $x^n - y^n$ is divisible by $x - y$, where x and y are any two integers with $x \neq y$.

SOLUTION:

1.Basis Step: For n = 1

$$x^n - y^n = x^1 - y^1 = x - y$$

which is clearly divisible by $x - y$. So, the statement is true for n = 1.

2.Inductive Step:

Suppose the statement is true for n = k, i.e.,

$x^k - y^k$ is divisible by $x - y$……………………(1)

We need to prove that $x^{k+1} - y^{k+1}$ is divisible by $x - y$

Now

$$x^{k+1} - y^{k+1} = x^k \cdot x - y^k \cdot y$$

$$= x^k \cdot x - x \cdot y^k + x \cdot y^k - y^k \cdot y \quad \text{(introducing } x.y^k\text{)}$$

$$= (x^k - y^k) \cdot x + y^k \cdot (x-y)$$

The first term on R.H.S=$(x^k - y^k)$ is divisible by $x - y$ by inductive hypothesis (1).

The second term contains a factor $(x-y)$ so is also divisible by $x - y$.

Thus $x^{k+1} - y^{k+1}$ is divisible by $x - y$. Hence, by mathematical induction $x^n - y^n$ is divisible by $x - y$ for any integer $n \geq 1$.

PROVING AN INEQUALITY:

Use mathematical induction to prove that for all integers $n \geq 3$.

$$2n + 1 < 2^n$$

SOLUTION:

1.Basis Step: For n = 3

L.H.S= 2(3) + 1 = 6 + 1 = 7

R.H.S = $2^3 = 8$

Since 7 < 8, so the statement is true for n = 3.

2.Inductive Step:

Suppose the statement is true for n =k, i.e.,

$2k + 1 < 2^k$…………………(1) $k \geq 3$

We need to show that the statement is true for n = k+1,

i.e.; $2(k+1) + 1 < 2^{k+1}$…………..(2)

Consider L.H.S of (2)

$= 2 (k+1) + 1$

$= 2k + 2 + 1$

$= (2k + 1) + 2$

$< 2^k + 2$ using (1)

$< 2^k + 2^k$ (since $2 < 2^k$ for $k \geq 3$)

$< 2 \cdot 2^k = 2^{k+1}$

Thus $2(k+1)+1 < 2k+1$ (proved)

EXERCISE:

Show by mathematical induction

$1 + n x \leq (1+x)^n$

for all real numbers x > - 1 and integers $n \geq 2$

SOLUTION:

1. Basis Step:

For n = 2

L.H.S = 1 + (2) x= 1 + 2x

RHS $= (1 + x)^2 = 1 + 2x + x^2 > 1 + 2x$ $(x^2 > 0)$

⇒ statement is true for n = 2.

2.Inductive Step:

Suppose the statement is true for n = k.

That is, for $k \geq 2$, $1 + k x \leq (1 + x)^k$………………..(1)

We want to show that the statement is also true for n = k + 1 i.e.,

$$1 + (k + 1)x \le (1 + x)^{k+1}$$

Since x > - 1, therefore 1 + x > 0.

Multiplying both sides of (1) by (1+x) we get

$(1+x)(1+x)^k \ge (1 + x)(1 + kx)$

$= 1 + kx + x + kx^2$

$= 1 + (k + 1)x + kx^2$

but

$$\left[\begin{array}{ll} x > -1, & \text{so } x^2 \ge 0 \\ \& \; k \ge 2, & \text{so } kx^2 \ge 0 \end{array}\right.$$

so

$(1+x)(1+x)^k \ge 1 + (k + 1)x$

Thus $1 + (k+1)x \le (1+x)^{k+1}$. Hence by mathematical induction, the inequality is true.

PROVING A PROPERTY OF A SEQUENCE:

Define a sequence $a_1, a_2, a_3, \ldots$ as follows:

$a_1 = 2$

$a_k = 5a_k-1$ for all integers $k \ge 2$ ....(1)

Use mathematical induction to show that the terms of the sequence satisfy the formula.

$a_n = 2 \cdot 5^{n-1}$ for all integers $n \ge 1$

SOLUTION:

1.Basis Step:

For n = 1, the formula gives

$a_1 = 2 \cdot 5^{1-1} = 2 \cdot 5^0 = 2 \cdot 1 = 2$

which confirms the definition of the sequence. Hence, the formula is true for n = 1.

2.Inductive Step:

Suppose, that the formula is true for n = k, i.e.,

$a_k = 2 \cdot 5^{k-1}$ for some integer $k \ge 1$

We show that the statement is also true for n = k + 1. i.e.,

$a_{k+1} = 2\cdot 5^{k+1-1} = 2\cdot 5^{k}$

Now

$a_{k+1} = 5\cdot a_{k+1-1}$ [by definition of $a_1, a_2, a_3 \ldots$ or by putting k+1 in (1)]

$= 5\cdot a_k$

$= 5\cdot(2\cdot 5^{k-1})$ by inductive hypothesis

$= 2\cdot(5\cdot 5^{k-1})$

$= 2\cdot 5^{k+1-1}$

$= 2\cdot 5^{k}$

which was required.

EXERCISE:

A sequence $d_1, d_2, d_3, \ldots$ is defined by letting $d_1 = 2$ and $d_k = \frac{d_{k-1}}{k}$

for all integers $k \geq 2$. Show that $d_n = \frac{2}{n!}$ for all integers $n \geq 1$, using mathematical induction.

SOLUTION:

1.Basis Step:

For n = 1, the formula $d_n = \frac{2}{n!}$; $n \geq 1$ gives

$$d_1 = \frac{2}{1!} = \frac{2}{1} = 2$$

which agrees with the definition of the sequence.

2.Inductive Step:

Suppose, the formula is true for n=k. i.e.,

$d_k = \frac{2}{k!}$ for some integer $k \geq 1$……………(1)

We must show that

$$d_{k+1} = \frac{2}{(k+1)!}$$

Now, by the definition of the sequence.

$$d_{k+1} = \frac{d_{(k+1)-1}}{(k+1)} = \frac{1}{(k+1)} d_k \qquad using\ d_k = \frac{d_{k-1}}{k}$$

$$= \frac{1}{(k+1)} \frac{2}{k!}$$

$$= \frac{2}{(k+1)!} \qquad \text{using (1)}$$

Hence the formula is also true for n = k + 1. Accordingly, the given formula defines all the terms of the sequence recursively.

EXERCISE:

Prove by mathematical induction that

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

Whenever n is a positive integer greater than 1.

SOLUTION:

1. Basis Step: for n = 2

L.H.S

$$= 1 + \frac{1}{4} = \frac{5}{4} = 1.25$$

R.H.S $= 2 - \frac{1}{2} = \frac{3}{2} = 1.5$

Clearly LHS < RHS

Hence the statement is true for n = 2.

2.Inductive Step:

Suppose that the statement is true for some integers k > 1, i.e.;

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} < 2 - \frac{1}{k} \qquad (1)$$

We need to show that the statement is true for n = k + 1. That is

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(k+1)^2}<2-\frac{1}{k+1} \qquad (2)$$

Consider the L.H. S of (2)

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(k+1)^2}=1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}+\frac{1}{(k+1)^2}$$

$$<\left(2-\frac{1}{k}\right)+\frac{1}{\left(k+1\right)^2}$$

$$=2-\left(\frac{1}{k}-\frac{1}{(k+1)^2}\right)$$

We need to prove that

$$2-\left(\frac{1}{k}-\frac{1}{(k+1)^2}\right)\le 2-\frac{1}{k+1}$$

or $$-\left(\frac{1}{k}-\frac{1}{(k+1)^2}\right)\le -\frac{1}{k+1}$$

or $$\frac{1}{k}-\frac{1}{(k+1)^2}\ge\frac{1}{k+1}$$

or $$\frac{1}{k}-\frac{1}{k+1}\ge\frac{1}{(k+1)^2}$$

Now $$\frac{1}{k}-\frac{1}{k+1}=\frac{k+1-k}{k(k+1)}$$

$$=\frac{1}{k(k+1)}>\frac{1}{(k+1)^2}$$

Definition If a proposition $P(n)$ is true for a positive **odd/even** integer s and that $P(k)$ is true implies $P(k+2)$ is also true, then $P(n)$ is true for all positive odd/even integers $n \geq s$.

Assignments:

Problem 1 Prove, by induction, that

$$\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{2n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...+\frac{1}{2n-1}-\frac{1}{2n}$$

for all positive integers *n*.

Problem 2 Given a sequence $u_1, u_2, \ldots, u_n, \ldots$ such that $u_1 = 1$ and $u_n = u_{n-1} + 3$, $(n \geq 2)$.

Show that $u_n = 3n - 2$, for all positive integers n.

Problem 3 Prove, by induction, that $n(n^2+5)$ is divisible by 6 for all positive integers *n*.

Problem 4 Prove, by mathematical induction, that $5^n - 2^n$ is divisible by 21 for all positive even integers *n*.

Problem 5 Prove, by mathematical induction, that $5^n - 3^n - 2^n$ is divisible by 30 for all positive odd integers *n* greater than 1.

Problem 6 A sequence of real numbers $a_0, a_1, \ldots, a_n, \ldots$ is defined by

$$a_0 = 1, a_1 = 7 \text{ and } a_{n+2} - 4a_{n+1} + 3a_n = 0 \text{ for } n = 0,1,2,\ldots$$

Prove, by induction, that $a_n = 3^{n+1} - 2$ for all non-negative integers n.

Ex. Find an explicit definition of the sequence defined recursively by
(i). $a_1 = 7, a_n = 2a_{n-1} + 1$ **for n ≥ 2.**
(ii). $a_1 = 4, a_n = a_{n-1} + n$ **for n ≥ 2.**

(i).By repeated use of the given recursive defn, we find that

$$a_n = 2a_{n-1} + 1 = 2(2a_{n-2} + 1) + 1$$
$$= 2[2(2a_{n-3} + 1) + 1] + 1 = 2^3 a_{n-3} + 2^2 + 2 + 1$$
....
$$= 2^{n-1} a_{n-(n-1)} + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1 \text{ (G.P)}$$
$$= 2^{n-1} a_1 + (2^{n-2} + 2^{n-3} + \ldots + 2 + 1).$$
$$= 7(2^{n-1}) + (2^{n-1} - 1) = 8(2^{n-1}) - 1.$$

The Fibonacci numbers defined recursively by
(i). $F_0 = 0, F_1 = 1$ **and**
$F_n = F_{n-1} + F_{n+1}$ **for n ≥ 2.**
Evaluate F_2 **to** F_{10}

Solution:

$F_2 = F_1 + F_0 = 1 + 0 = 1$ $F_3 = F_2 + F_1 = 1 + 1 = 2$
$F_4 = F_3 + F_2 = 2 + 1 = 3$ $F_5 = F_4 + F_3 = 3 + 2 = 5$
$F_6 = F_5 + F_4 = 5 + 3 = 8$ $F_7 = F_6 + F_5 = 8 + 5 = 13$
$F_8 = F_7 + F_6 = 13 + 8 = 21$ $F_9 = F_8 + F_7 = 21 + 13 = 34$
$F_{10} = F_9 + F_8 = 34 + 21 = 55$

IF $F_0, F_1, F_2, ...$ Fibonacci numbers, Prove that $\sum_{i=1}^{n} F_i^2 = F_n X F_{n+1}$ for all positive integers n.

Solution: we first note that $\sum_{i=1}^{1} F^2_i = F^2_0 + F_1^2 = 0+1=1=1\times1=F_1\times F_2$

because, $F_1=F_2=1$.

This verifies the result n=1.

Next, we assume the result n=k+1.

$\sum_{i=1}^{k} F_i^2 = F_k X F_{k+1}$ consequently,

$$\sum_{i=1}^{k+1} F_i^2 = \sum_{i=1}^{k} F_i^2 + F^2_{k+1} = (F_k \times F_{k+1}) + F^2_{k+1}$$

$$= F_{k+1} \times (F_k + F_{k+1})$$

$$= F_{k+1} \times F_{k+2} \text{ because } F_{k+2} = F_{k+1} \times F_k$$

This shows that the result is true for n=k+1.

3.

The Lucas numbers are defined recursively by $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Evaluate L_2 to L_{10}

$Sol: L_2 = L_1 + L_0$ =1+2=3 $L_3 = L_2 + L_1$ =3+1=4

$L_4 = L_3 + L_2$ =4+3=7 $L_5 = L_4 + L_3$ =7+4=11

$L_6 = L_5 + L_4$ =11+7=18 $L_7 = L_6 + L_5$ =18+11=29

$L_8 = L_7 + L_6$ =29+18=47 $L_9 = L_8 + L_7$ =47+29=76

$L_{10} = L_9 + L_8$ =76+47=123

The sequence formed Lucas no. is Lucas sequence.

If F_i 's are Fibonacci numbers and The L_i's are Lucas numbers are Prove that $5F_{n+2} = L_{n+4} + L_n$ for $n \geq 0$.

Sol : For n = 0, and n = 1

$L_4 - L_0 = 5F_2$, $L_5 - L_1 = 5F_3$ Since

$L_0 = 2$, $L_1 = 1$, $L_4 = 7$ $L_5 = 11$ and $F_2 = 1$, $F_3 = 2$

these are true.

We assume that the result is true for n = 0,1,...,k where k ≥ 1.

Then, we find that

$$L_{k+5} - L_{k+1} = (L_{k+4} + L_{k+3}) - (L_k + L_{k-1})$$
$$= (L_{k+4} - L_k) + (L_{k+3} - L_{k-1})$$
$$= (L_{k+4} - L_k) + (L_{(k-1)+4} - L_{k-1})$$
$$= 5F_{k+2} + 5F_{k+1} = 5F_{k+3}$$

This shows that required result is true for n=k+1.

For integers m and k, the Eulerian numbers $a_{m,k}$ are defined recursively as follows:

$$a_{0,0} = 1; a_{m,k} = 0 \text{ for } k \geq m > 0,\ a_{m,k} = 0 \text{ for } k < 0$$

$$a_{m,k} = (m-k)a_{m-1,k-1} + (k+1)a_{m-1,k} \text{ for } 0 \leq k \leq m-1$$

Determine the values of $a_{m,k}$ for $1 \leq m \leq 5, 0 \leq k \leq m-1$.

Sol : For $0 \leq k \leq m-1$, we have by definition

$a_{m,k} = (m-k)a_{m-1,k-1} + (k+1)a_{m-1,k}$ ....(1)

when m = 1, this expression holds for k=0,1 we get

$a_{1,0} = (1-0)a_{0,-1} + (0+1)a_{0,0} = 1$

Because $a_{0,-1} = 0$ and $a_{0,0} = 1$

when m = 2, this expression holds for k=0,1 we get

$a_{2,0} = (2-0)a_{1,-1} + (0+1)a_{1,0} = 1$

$a_{2,1} = (2-1)a_{1,0} + (1+1)a_{1,1} = 1 + 0 = 1$

when m = 3, this expression holds for k=2,1,0 we get

$a_{3,2} = (3-2)a_{2,1} + (2+1)a_{2,2} = 1 + 0 = 1$(using definition)

$a_{3,1} = (3-1)a_{2,0} + (1+1)a_{2,1} = 2(1) + 2(1) = 4$

$a_{3,0} = (3-0)a_{2,-1} + (0+1)a_{2,0} = 0 + 1 = 1$

similaly, weget

$a_{4,3} = 1, a_{4,2} = 11, a_{4,1} = 11, a_{4,0} = 1,$

$a_{5,4} = 1, a_{5,3} = 26, a_{5,2} = 66, a_{5,1} = 26, a_{5,0} = 1.$

The Ackermann's numbers $A_{m,n}$ *are* defined recursively for $m, n \in N$ as follows:

$A_{0,n} = n + 1$ for $n \geq 0$,

$A_{m,0} = A_{m-1,1}$ for $m \geq 0$,

$A_{m,n} = A_{m-1,p}$ where $p = A_{m,n-1}$ for $m, n > 0$

prove that $A_{1,n} = n + 2$ for all $n \in N$.

Soln: we first note that $A_{1,0}=A_{0,1}=1+1=2=0+2$(II & I steps of Def)

This verifies the result n=0.

We assume that the result is true for $n=k \geq 0$

we asume $A_{1,k} = k + 2$ for $k \geq 0$. Then we find

$$A_{1,k+1} = A_{0,p} \text{ where } p=A_{1,k} (IIIstep)$$

$$= A_{0,k+2} = (k+2)+1 = (k+1)+2 (because\ p=A_{1,k} = k+2).$$

This shows that the required result is true for n=k+1.

Well formed formulas

In the study of logic, the following are called well-formed formulas

- Components of a compound proposition.
- The tautology T_0
- The Contradiction F_0
- ˜¬Where P and Q are themselves well formed formulas.

Problem :

If p, q, r are primitive statements, prove that the following are well formed formulas.

1. $(p \vee q) \rightarrow T_0 \wedge(\neg r)$, 2. $(p \wedge (\neg r)) \leftrightarrow (r \vee F_0)$

Proof :

(1). Since p and q are primitive statements, $p \vee q$ is a well formed formula. Also, since r is a primitive statement, $\neg r$ is a well-formed formula. Consequently $T_0 \wedge(\neg r)$ is a well-formed formula.

Finally, since $p \vee q$ is a well-formed formula and $T_0\wedge(\neg r)$ is a well formed formula, so is $(p \vee q)\rightarrow T_0 \wedge(\neg r)$

(2), since q is a primitive statement, $\neg q$ is a well formed formula. Since p is a primitive statement, $p\wedge(\neg r)$ is a Well formed formula. Also, since r is a primitive statement $r \vee F_0$ is a well formed formula.

Consequently $\neg(r \vee F_0)$is a well formed formula.

Finally, since $p \wedge (\neg r)$ is a well formed formula and $\neg(r \vee F_0)$ is a well formed formula, so is $(p \wedge (\neg r)) \leftrightarrow (r \vee F_0)$

Another example:

The well-formed formulae of variables, numerals and operators from {+, -, *, /, ^} are defined by:

x is a well-formed formula if x is a numeral or variable.

(f + g), (f – g), (f * g), (f / g), (f ^ g) are well-formed formulae if f and g are.

An algorithm is called recursive if it solves a problem by reducing it to an instance of the same problem with smaller input.

With this definition, we can construct formulae such as:

(x – y)

((z / 3) – y)

((z / 3) – (6 + 5))

((z / (2 * 4)) – (6 + 5))

Example I: Recursive Euclidean Algorithm

An algorithm is called recursive if it solves a problem by reducing it to an instance of the same problem with smaller input.

Example I: Recursive Euclidean Algorithm

```
        procedure gcd(a, b: nonnegative integers with a < b)
        if a = 0 then gcd(a, b) := b
        else gcd(a, b) := gcd(b mod a, a)
```

Example II: Recursive Fibonacci Algorithm

```
        procedure fibo(n: nonnegative integer)
        if n = 0 then fibo(0) := 0
        else if n = 1 then fibo(1) := 1
        else fibo(n) := fibo(n – 1) + fibo(n – 2)
procedure iterative_fibo(n: nonnegative integer)
        f n = 0 then y := 0
else
begin
        x := 0
        y := 1
        for i := 1 to n-1
        begin
                z := x + y
                x : = y
                y := z
        end
end   {y is the n-th Fibonacci number}
```

For every recursive algorithm, there is an equivalent iterative algorithm.

Recursive algorithms are often shorter, more elegant, and easier to understand than their iterative counterparts.

However, iterative algorithms are usually more efficient in their use of space and time

Recursive Definitions-Applications

The usual way of defining a set is to specify some property satisfied by all its elements.

For example, an integer may have the property of being *even*. And so we can talk about the set of all multiples of 2 (*even numbers*) :

$$\{n \mid n = 2k \text{ for some } k \in \mathbb{Z}\}$$

The set is defined as the collection of all elements in our universal set *that have a certain property.*

The drawback of this kind of definition is that it does not give any information about how to find, or build, the elements of the set.

We will see how to generate the elements of a set. This is called *definition by recursion.*

- In Chemistry, every chemical element is a different kind of atom, with a different number and arrangement of electrons. Chemical compounds consist of combinations of elements.
- Not every combination is possible — the allowable combinations depend on the arrangements of electrons in the atoms.
- This is a nonmathematical example of the pattern that we find in all recursive definitions:

we have one or more basic building blocks, or atoms, andwe have one or more permissible ways to combine the atoms in order to generate more complex things.

The Set of Even Integers

- To give a recursive definition of the set of even integers, think about how to generate even integers from the 'simplest' integer, namely 0.
- Even integers are either positive or negative. The positive even integers can be generated by adding 2 to 0 one or more times to get 2,*4,6,8,…*
- The negative even integers can be generated by subtracting 2 from 0.
- So the basic strategy of our recursive definition of the set of even integers consists of two steps

    1. to specify an atom, namely 0, and
    2. to say how members of the set are to be generated from the atom 0.

namely if *k is the* atom or some number already generated from the atom, then $k + 2$ *and* $k - 2$ *are allowable* numbers to generate.

The Factorial Function

- Let $\mathbb{Z}^+ = \{0, 1, 2, 3, \ldots\}$ *be the set of nonnegative integers. Let* $f : \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ *be the factorial function. Then*
    - $f(0) = 1$ *and*
    - $f(n + 1) = (n + 1) \times f(n)$.

- The *factorial function is an example of a function* having a recursive definition.
- The basic building block of the factorial function tells us that on the input 0 the function outputs $f(0) = 1$.
- *More generally, the function can use the* input $n \in \mathbb{Z}^+$, *to calculate the output for* the next input, $n + 1 \in \mathbb{Z}^+$, *and* should set $f(n + 1) = (n + 1) \times f(n)$.
- This tells us how to work out, say, $f(3)$. *We know*

$$f(0) = 1, \text{ so } f(3) = 3 \times f(2) = 3 \times 2 \times f(1) = 3 \times 2 \times 1 \times f(0) = 6.$$

- *Our definition gave us the atom from* which to start, and told us how to generate the rest of the function.

The Parenthesis Language:

- A set of strings over an alphabet is called a formal language. The most important examples are programming languages for computers.
- Here is a very simple (and not very useful) example of a formal language over the alphabet $A = \{(,)\}$. We call the language *P for Parenthesis.*
- As atom take the empty string λ. *Thus* $\lambda \in P$. Next, if *x is already a string in P, then* $(x) \in P$ also.
  - The strings of the language are

$$\lambda, (), (()), ((())), \ldots$$

- In other words, we generate longer strings by adding more parentheses on either side.
- Our definition has again started by giving a building block and continued by describing how to use previously generated things to build new things. So it is a recursive definition.

There are actually *two parts to every recursive* definition:

1. We must give one or more atoms as a starting point.
2. We must say how previously generated items can be used to build new items.

We must keep in mind that :

- The *only items in* the recursively generated set are those which can be built from the atoms in a finite number of steps.
- Nothing else gets to be a member of the recursively defined set.

Thus the binary numerals are :

1, *10, 11, 100, 101, 110, 111, 1000, 1001, 1010,…*

of course, we know that the binary numeral 1 represents the number one, 10 represents two, 11 represents three, and so on…

For example the binary numeral 1010 stands for the number that, in the usual decimal notation, would be calculated as

$$1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$$

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